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Approximation for Definite Integrals of Bivariate Functions through Second-Degree Taylor Polynomials

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Abstract:

In practical applications of Mathematics, numerical methods are extensively utilized across various domains. Numerical integration, a vital component of numerical analysis, involves approximating definite integrals. Obtaining exact results for definite integrals of functions with two variables is often challenging in real-world scenarios. Consequently, computational scientists focus on approximation techniques to mitigate this complexity. Numerous studies exist in the literature addressing this issue. This paper proposes an approach for approximating definite integrals of functions with two variables using second-degree Taylor polynomials to minimize errors. We developed and applied this approximation method to selected known functions and compare the approximated values with exact values of the integrals. Finally, we compared the outcomes of our approach with well-known numerical techniques like tangent plane approximation and Simpson's 1/3 rule. Our findings indicate that proposed method yields superior results compared to the other two approaches considered.

Keywords: definite integrals, numerical integration, functions of two variables, Taylor polynomial approximation

1. Introduction

Numerical integration is a branch of study concerned with determining the approximate numerical value of a definite integral and crucial for computing specified integrals in numerical analysis. The fundamental objective of numerical integration is to compute an approximate solution to a defined integral. This process differs from analytical integration in two key ways: it involves error analysis since numerical integration provides an approximation that may not yield precise results, and it focuses solely on generating a numerical result that approximates the area under a curve. Numerical integration is pivotal across various domains such as mathematics, engineering, physical science, and computer science.

Previous studies have explored various approaches to numerical integration. One of the earliest and most widely used numerical integration techniques is Simpson's rule [1]. Introduced by Thomas Simpson in the 18th century, this method approximates the integral of a function by fitting parabolic arcs to small intervals. However, numerical integration remains a vibrant area of research, with a rich history of development and ongoing innovation. By leveraging a diverse array of methods and techniques, researchers continue to push the boundaries of accuracy, efficiency, and applicability in numerical integration. Recently, Peer [2] delved into tangent plane approximation of definite integrals, while Hisam and Faham [3] derived a formula for line integrals using the second-degree Taylor polynomial. Building upon these foundations, this research extends these methodologies to evaluate definite integrals of functions of two variables.

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Consider a surface S represented by the equation $z = f(x, y)$, as illustrated in Figure 1 below. The region R corresponds to the projection of this surface onto the xy-plane. Our objective is to determine the volume of the surface S, which lies above R and below the graph of the positive function f [4].

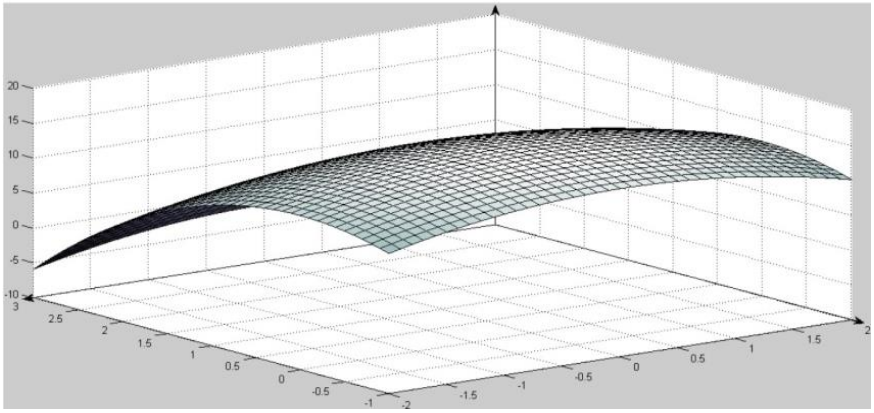

Figure 1: Three-dimensional graph of a surface $f(x, y)$

To compute this volume, we partition the rectangle R into sub-rectangles. We divide the interval $[a, b]$ into m sub-rectangles$[x_i,\ x_{i+1}]; i = 1, 2, \cdots, m$ of equal width $\Delta x = \frac{b-a}{m}$ and divide $[c, d]$ into n sub-rectangles $[y_j,\ y_{j+1}]; j = 1, 2, \cdots, n$ of equal width $\Delta y = \frac{d-c}{n}$.

Then, we compute the volume beneath f by double integrating over the region R as follows: for sample points (x_{ij}^*, y_{ij}^*) in$[x_i,\ x_{i+1}] \times [y_j,\ y_{j+1}]$ and $\Delta A = \Delta x \Delta y$,

$$\iint_R f(x, y)dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A \qquad (1)$$

2. Methodology

In this approach, we employ the Taylor polynomial of degree two to approximate the given function $f(x, y)$ of two variables across each sub-rectangle. The crux of this method involves selecting the midpoint of each rectangle as the center for our approximation. By leveraging the second-degree Taylor polynomial, we aim to capture the local behavior of the function $f(x, y)$ within each sub-region, thereby facilitating a more accurate approximation across the entire domain. This strategy enables us to intricately dissect the function's behavior within each sub-rectangle, allowing for a finer-grained approximation that better aligns with the function's curvature and variation. Ultimately, by judiciously choosing the midpoint of each rectangle as the focal point for our approximation, we endeavor to enhance the precision and fidelity of our numerical integration technique.

Now we define an approximation function $P_2(x, y)$ for $f(x, y)$ on the sub-rectangle $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$, by taking $\bar{x}_i = x_0 + \frac{(2i-1)\Delta x}{2}$, $\bar{y}_j = y_0 + \frac{(2j-1)\Delta y}{2}$ as midpoints of $[x_i, x_{i+1}], [y_j, y_{j+1}]$ respctively, as

$$P_2(x,y) = f(\bar{x}_i,\bar{y}_j) + f_x(\bar{x}_i,\bar{y}_j)(x-\bar{x}_i) + f_y(\bar{x}_i,\bar{y}_j)(y-\bar{y}_j) + \frac{f_{xx}(\bar{x}_i,\bar{y}_j)(x-\bar{x}_i)^2}{2} + f_{xy}(\bar{x}_i,\bar{y}_j)(x-\bar{x}_i)(y-\bar{y}_j) + \frac{f_{yy}(\bar{x}_i,\bar{y}_j)(y-\bar{y}_j)^2}{2}. \quad (2)$$

Then, the surface integral of a particular sub-rectangle $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$ is given by,

$$\int_{y_j}^{y_{j+1}} \int_{x_i}^{x_{i+1}} P_2(x,y)\, dxdy. \quad (3)$$

To evaluate the double integral over the region *R*, we sum up all sub-rectangles in *R* and then substituting $\bar{x}_i = \frac{1}{2}(x_{i+1} + x_i)$, $\bar{y}_i = \frac{1}{2}(y_{j+1} + y_j)$, $x_{i+1} - x_i = \Delta x$ and $y_{j+1} - y_j = \Delta y$, in (2), we get equation (3) in the form,

$$\sum_{i=1}^{m}\sum_{j=1}^{n} \int_{y_j}^{y_{j+1}} \int_{x_i}^{x_{i+1}} P_2(x,y)\, dxdy = \sum_{i=1}^{m}\sum_{j=1}^{n} \Delta x\Delta y \left[f(\bar{x}_i,\bar{y}_j) + \frac{1}{24} f_{xx}(\bar{x}_i,\bar{y}_j)(\Delta x)^2 + \frac{1}{24} f_{yy}(\bar{x}_i,\bar{y}_j)(\Delta y)^2 \right].$$

Thus,

$$\int_c^d \int_a^b f(x,y)\, dxdy \approx \sum_{i=1}^{m}\sum_{j=1}^{n} \Delta x\Delta y \left[f(\bar{x}_i,\bar{y}_j) + \frac{1}{24} f_{xx}(\bar{x}_i,\bar{y}_j)(\Delta x)^2 + \frac{1}{24} f_{yy}(\bar{x}_i,\bar{y}_j)(\Delta y)^2 \right] \quad (4)$$

3. Results and Discussion

In this section, we intend to present the outcomes of our novel approach utilizing second-degree Taylor polynomial approximation in comparison to the traditional tangent plane approximation (T.P.A) and Simpson's 1/3 rule. We conduct our analysis by progressively increasing the values of m and n, the number of sub-rectangles, to observe the corresponding improvements in accuracy.

The computations were efficiently carried out using the mathematical software MATLAB, facilitating rapid execution and analysis of our proposed method alongside the established techniques. As m and n increase, each approximation exhibits enhanced accuracy, reflecting the intrinsic relationship between finer partitioning and improved precision in numerical integration. Notably, our proposed second-degree Taylor polynomial approximation showcases a marked reduction in error with increasing numbers of sub-rectangles. This phenomenon indicates the

superior performance of our method, particularly evident when dealing with a smaller number of sub-rectangles. The efficacy of our approach lies in its ability to capture the local behavior of the function more accurately, leading to more precise integral approximations. See table 1 and Table 2 for further details.

Furthermore, our analysis reveals a noteworthy observation regarding Simpson's 1/3 rule: when m and n are chosen as even numbers, this method exhibits diminished performance. This phenomenon underscores the sensitivity of Simpson's 1/3 rule to the choice of intervals, highlighting the importance of considering the parity of m and n for optimal performance.

4. Conslusion

This paper introduces a novel formula for obtaining numerical approximations of definite surface integrals of the form $\iint_R f(x,y)\,dxdy$, where R is a region in the xy-plane bounded by the intervals $[a,b]$ and $[c,d]$. Our proposed approach, imposing the second-degree Taylor polynomial approximation, demonstrates notable advantages over existing methods, particularly in terms of efficiency and accuracy. In the realm of computational scientists, where efficiency is a critical criterion, our proposed formula emerges as a valuable tool. Its ability to deliver more accurate results, especially when compared to certain existing methods. As we continue to advance computational techniques, our proposed formula contributes to the ongoing efforts to refine and optimize numerical integration methodologies, providing a valuable resource for researchers and practitioners across various domains.

5. References

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Integral	Exact Value	Rule	For m = n = 2		For m = n = 5		For m = n = 8		For m = n = 11	
			Value	Error	Value	Error	Value	Error	Value	Error
$\int_1^2 \int_0^1 (x^2y + xy^2)\, dydx$	1.6667	T.P.A	1.3125	0.3542	1.6040	0.0627	1.6416	0.0251	1.6533	0.0134
		Simpson	1.6667	3.3333e-05	1.3056	0.3611	1.6667	3.3333e-05	1.4852	0.1815
		Proposed	1.6667	3.3333e-05	1.6667	3.3333e-05	1.6667	3.3333e-05	1.6667	3.3333e-05
$\int_1^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \sin(x+y)\, dydx$	2.0000	T.P.A	2.6588	0.6588	2.1136	0.1136	2.0447	0.0447	2.0237	0.0237
		Simpson	2.0091	0.0091	1.7611	0.2389	2.0000	3.3182e-05	1.8983	0.1017
		Proposed	1.9978	0.0022	1.9999	5.4467e-05	2.0000	8.2784e-06	2.0000	2.3132e-06
$\int_0^1 \int_0^1 e^{(x+y)}\, dydx$	2.9525	T.P.A	2.6309	0.3216	2.8911	0.0614	2.9274	0.0251	2.9390	0.0135
		Simpson	2.9545	0.0020	2.4185	0.5340	2.9525	4.3629e-07	2.6885	0.2640
		Proposed	2.9520	5.1184e-04	2.9525	2.0646e-05	2.9525	9.5582e-06	2.9525	8.1178e-06

Table 1 comparison of tangent plane approximation, Simpson's $\frac{1}{3}$ rule and proposed rule for m = n = 2, m = n = 5, m = n = 8 and m = n = 11.

Integral	Exact Value	Rule	Error			
			For m = n = 25	For m = n = 50	For m = n = 125	For m = n = 200
$\int_1^2 \int_0^1 (x^2y + xy^2)\, dydx$	1.6667	T.P.A	0.0027	6.9600e-04	1.3974e-04	7.4938e-05
		Simpson	0.0836	3.3333e-05	0.0172	3.3333e-05
		Proposed	3.3333e-05	3.3333e-05	3.3333e-05	3.3333e-05
$\int_1^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \sin(x+y)\, dydx$	2.0000	T.P.A	0.0046	0.0012	1.8423e-04	7.1965e-05
		Simpson	0.0432	2.1649e-08	0.0084	8.4659e-11
		Proposed	8.6608e-08	5.4120e-09	1.3871e-10	2.1077e-11
$\int_0^1 \int_0^1 e^{(x+y)}\, dydx$	2.9525	T.P.A	0.0027	6.8864e-04	1.1728e-04	5.0492e-05
		Simpson	0.1208	7.5527e-06	0.0248	7.5580e-06
		Proposed	7.5790e-06	7.5593e-06	7.5580e-06	7.5580e-06

Table 2 comparison of tangent plane approximation, Simpson's $\frac{1}{3}$ rule and proposed rule for m = n = 25, m = n = 50, m = n = 125 and m = n = 20